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NOTE

ON A DEGREE PROPERTY OF CROSS-INTERSECTING FAMILIES

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Motivated by some applications in computational complexity, Razborov and Vereshchagin proved a degree bound for cross-intersecting families in [1]. We sharpen this result and show that our bound is best possible by constructing appropriate families. We also consider the case of cross-t-intersecting families.

Let X be a finite set, and $2^X=\{A:A\subset X\}$ its power set. For an integer $r\geq 0$ we set $\binom{X}{r}=\{F\in 2^X : |F|=r\}$ and $\binom{X}{\leq r}=\{F\in 2^X : |F|\leq r\}$. Given $\mathcal{F}\subset 2^X$ and $x\in X$ we define the degree of x in $\mathcal{F}$ as $\deg_{\mathcal{F}} x=|\{F\in\mathcal{F}:x\in F\}|$. Families $\mathcal{A}\subset 2^X$ and $\mathcal{B}\subset 2^X$ are called cross-t-intersecting, if $|A\cap B|\geq t$ for any $A\in\mathcal{A}$ and $B\in\mathcal{B}$. In case $t=1$ the families are called cross-intersecting. Razborov and Vereshchagin gave a probabilistic proof of the following

Theorem 1 **([1]).** *Let $\mathcal{A}\subset\binom{X}{\leq m}$ and $\mathcal{B}\subset\binom{X}{\leq n}$ be cross-intersecting families. Then there is an element $x\in X$ such that*

$$\deg_{\mathcal{A}} x \geq \frac{|\mathcal{A}|}{2n} \quad \textit{and} \quad \deg_{\mathcal{B}} x \geq \frac{|\mathcal{B}|}{2m}. \qquad \blacksquare$$

Slightly deepening ideas of their proof we get

Theorem 2. *Let $m,n>1$ be integers and $\mathcal{A}\subset\binom{X}{\leq m}$ and $\mathcal{B}\subset\binom{X}{\leq n}$ be cross-intersecting families. Then there is an element $x\in X$ such that*

$$\deg_{\mathcal{A}} x \geq \frac{|\mathcal{A}|}{2(n-1)} \quad \textit{and} \quad \deg_{\mathcal{B}} x \geq \frac{|\mathcal{B}|}{2(m-1)}. \tag{1}$$

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The inequalities are best possible: for every $m, n > 1$ *there are cross-intersecting families* $\mathcal{A} \subset \binom{X}{\le m}$ *and* $\mathcal{B} \subset \binom{X}{\le n}$ *with no* $x \in X$ *of*

$$\deg_{\mathcal{A}} x > \frac{|\mathcal{A}|}{2(n-1)} \quad \text{and} \quad \deg_{\mathcal{B}} x > \frac{|\mathcal{B}|}{2(m-1)}.$$

Proof. At first, observe the following simple properties of cross-intersecting families

$$\sum_{x \in A} \deg_{\mathcal{B}} x = \sum_{B \in \mathcal{B}} |A \cap B| \ge |\mathcal{B}| \text{ for any } A \in \mathcal{A}, \tag{2}$$

$$\sum_{x \in X} \deg_{\mathcal{A}} x \deg_{\mathcal{B}} x = \sum_{(A,B) \in \mathcal{A} \times \mathcal{B}} |A \cap B| \ge |\mathcal{A}||\mathcal{B}|. \tag{3}$$

Now, on the contrary to the statement, suppose there is not an element, satisfying (1). Consider the following partition of the ground set $X = X_1 \dot{\cup} X_2$, where

$$X_1 = \left\{ x \in X : \deg_{\mathcal{A}} x \ge \frac{|\mathcal{A}|}{2(n-1)} \right\}$$

and

$$X_2 = \left\{ x \in X : \deg_{\mathcal{A}} x < \frac{|\mathcal{A}|}{2(n-1)} \right\}.$$

Obviously, $\deg_{\mathcal{B}} x < \frac{|\mathcal{B}|}{2(m-1)}$ for every $x \in X_1$. Observe that

$$\sum_{x \in X_1} \deg_{\mathcal{A}} x \le (m-1)|\mathcal{A}| \quad \text{and} \quad \sum_{x \in X_2} \deg_{\mathcal{B}} x \le (n-1)|\mathcal{B}|. \tag{4}$$

Indeed, by (2) for any $A \in \mathcal{A}$ there is an $x \in A$ with $\deg_{\mathcal{B}} x \ge \frac{|\mathcal{B}|}{m} \ge \frac{|\mathcal{B}|}{2(m-1)}$, implying $A \cap X_2 \ne \varnothing$ and therefore $\sum_{x \in X_2} \deg_{\mathcal{A}} x \ge |\mathcal{A}|$. Consequently,

$$\sum_{x \in X_1} \deg_{\mathcal{A}} x = \sum_{x \in X} \deg_{\mathcal{A}} x - \sum_{x \in X_2} \deg_{\mathcal{A}} x \le m|\mathcal{A}| - |\mathcal{A}| = (m-1)|\mathcal{A}|.$$

The second inequality of (4) can be proved in the same way. Finally, using (3) and (4) we have

$$|\mathcal{A}||\mathcal{B}| \le \sum_{x \in X} \deg_{\mathcal{A}} x \deg_{\mathcal{B}} x = \sum_{x \in X_1} \deg_{\mathcal{A}} x \deg_{\mathcal{B}} x + \sum_{x \in X_2} \deg_{\mathcal{A}} x \deg_{\mathcal{B}} x$$

$$< \frac{|\mathcal{B}|}{2(m-1)} \sum_{x \in X_1} \deg_{\mathcal{A}} x + \frac{|\mathcal{A}|}{2(n-1)} \sum_{x \in X_2} \deg_{\mathcal{B}} x \le$$

$$\frac{|\mathcal{B}|}{2(m-1)}(m-1)|\mathcal{A}| + \frac{|\mathcal{A}|}{2(n-1)}(n-1)|\mathcal{B}| = |\mathcal{A}||\mathcal{B}|,$$

a contradiction, which proves (1). Given further Construction 1 shows that (1) is best possible. ∎

Theorem 2 can be generalized to

Theorem 3. *Let $t<m\le n$ be integers and either t be even or $t\le \frac{2m-1}{3}$ be odd. Suppose $\mathcal{A}\subset\binom{X}{\le m}$ and $\mathcal{B}\subset\binom{X}{\le n}$ are cross-t-intersecting families. Then there is an element $x\in X$ such that*[1]

$$\deg_{\mathcal{A}} x \ge \frac{t|\mathcal{A}|}{2(n-\lceil\frac{t}{2}\rceil)} \quad \text{and} \quad \deg_{\mathcal{B}} x \ge \frac{t|\mathcal{B}|}{2(m-\lceil\frac{t}{2}\rceil)}. \tag{5}$$

Proof. Assume, contrary to the statement, that there is not such an element. Partition X into $X_1\dot\cup X_2$, where

$$X_1=\left\{x\in X:\deg_{\mathcal{A}} x\ge \frac{t|\mathcal{A}|}{2(n-\lceil\frac{t}{2}\rceil)}\right\}$$

and

$$X_2=\left\{x\in X:\deg_{\mathcal{A}} x< \frac{t|\mathcal{A}|}{2(n-\lceil\frac{t}{2}\rceil)}\right\}.$$

We first show that $|A\cap X_2|\ge\lceil\frac{t}{2}\rceil$ for any $A\in\mathcal{A}$. Suppose the opposite, namely, that there is an $A'\in\mathcal{A}$ s.t. $|A'\cap X_2|=s<\lceil\frac{t}{2}\rceil$. Then

$$\begin{aligned} t|\mathcal{B}| &\le \sum_{B\in\mathcal{B}}|A'\cap B| = \sum_{x\in A'}\deg_{\mathcal{B}} x = \sum_{x\in A'\cap X_1}\deg_{\mathcal{B}} x + \sum_{x\in A'\cap X_2}\deg_{\mathcal{B}} x \\ &< (m-s)\frac{t|\mathcal{B}|}{2(m-\lceil\frac{t}{2}\rceil)} + \sum_{x\in A'\cap X_2}\deg_{\mathcal{B}} x. \end{aligned}$$

Thus, there is an $x'\in A'\cap X_2$ s.t.

$$\deg_{\mathcal{B}} x' > |\mathcal{B}|\frac{t(m-2\lceil\frac{t}{2}\rceil+s)}{s(2m-2\lceil\frac{t}{2}\rceil)} \ge |\mathcal{B}|,$$

a contradiction. Analogously, $|B\cap X_1|\ge\lceil\frac{t}{2}\rceil$ for any $B\in\mathcal{B}$. The rest of the proof runs as for Theorem 1. ∎

Remark. If
(i) t is even and $m,n>t$
or
(ii) $t\le\frac{2m-1}{3}$ is odd, and $m-\lceil\frac{t}{2}\rceil$ and $n-\lceil\frac{t}{2}\rceil$ are divisible by t,

[1] $\lceil x\rceil$ is the smallest integer grater than or equal to x.

then inequalities (5) are best possible. Namely, for such m,n,t there are cross-t-intersecting families $\mathcal{A}\subset\binom{X}{\le m}$ and $\mathcal{B}\subset\binom{X}{\le n}$ with no $x\in X$ of

$$\deg_{\mathcal{A}} x > \frac{t|\mathcal{A}|}{2(n-\lceil\frac{t}{2}\rceil)} \quad\text{and}\quad \deg_{\mathcal{B}} x > \frac{t|\mathcal{B}|}{2(m-\lceil\frac{t}{2}\rceil)},$$

as Constructions 2 and 3 show. ∎

The following constructions show that the bounds of Theorem 2 and 3 are best possible. To simplify notations, we set $[n]=\{1,\dots,n\}$ and $[k,n]=\{k,k+1,\dots,n\}$. We identify $2^{[n]}$ with $\{0,1\}^n$ (the set of all (0,1)-sequences of length n) via the indicator function. A family $\mathcal{F}\subset 2^{[n]}$ we write as $|\mathcal{F}|\times n$ (0,1)-matrix with the sets corresponding to rows. For integer $k,r\ge 1$, by I_k and $J_{k\times r}$ are denoted the $k\times k$ identity and $k\times r$ all-one matrices, resp. We set $\bar{M}$ for the complement matrix of a (0,1)-matrix M, i.e. $\bar{M}=J-M$. By M_1M_2 is denoted the concatenation of matrices M_1 and M_2.

Construction 1, $t=1$.
Fix any $m,n>1$. Let $\mathcal{A}=\mathcal{A}_1\bar{\mathcal{A}}_1 I_{2n-2}$, where $\mathcal{A}_1$ is $(2n-2)\times(m-1)$ matrix, whose columns are (0,1)-vectors of length $2n-2$ and weight $n-1$. Let $\mathcal{B}=I_{2m-2}(\bar{\mathcal{A}}_1\mathcal{A}_1)^T$. It is easy to see that $\mathcal{A}\subset\binom{[2m+2n-4]}{m}$ and $\mathcal{B}\subset\binom{[2m+2n-4]}{n}$ are cross-intersecting and for any $x\in[2m-2]$ $\deg_{\mathcal{B}} x=1=\frac{|\mathcal{B}|}{2m-2}$, while for any $x\in[2m-1,2m+2n-4]$ $\deg_{\mathcal{A}} x=1=\frac{|\mathcal{A}|}{2n-2}$.

Construction 2, $t=2k-1\le\frac{2m-1}{3}(k\ge 2)$.
Choose integers $m,n>t$ s.t. $m+k-1$ and $n+k-1$ are divisible by $t=2k-1$. Set $m'=\frac{m+k-1}{2k-1}$ and $n'=\frac{n+k-1}{2k-1}$. Let $\mathcal{A}'$ and $\mathcal{B}'$ be constructed by the previous construction for m' and n'. To get required families $\mathcal{A}$ and $\mathcal{B}$, replace the entries of $\mathcal{A}'$ and $\mathcal{B}'$ in the following way: $1\to\underbrace{11\dots11}_{k}$, $0\to\underbrace{00\dots00}_{k}$ in the identity matrices of $\mathcal{A}'$ and $\mathcal{B}'$, and the remaining $0\to\underbrace{1\dots101\dots1}_{k}$ with a 0 in any (not fixed) position. $\mathcal{A}\subset\binom{[2k(m'+n'-2)]}{m}$ with $|\mathcal{A}|=2(n'-1)$ and $\mathcal{B}\subset\binom{[2k(m'+n'-2)]}{n}$ with $|\mathcal{B}|=2(m'-1)$ are cross-$(2k-1)$-intersecting,and for any $x\in[2k(m'-1)]$ $\deg_{\mathcal{B}} x=1=\frac{(2k-1)|\mathcal{B}|}{2(m-k)}$, while for any $x\in[2k(m'-1)+1,2k(m'+n'-2)]$ $\deg_{\mathcal{A}} x=1=\frac{(2k-1)|\mathcal{A}|}{2(n-k)}$.

Construction 3, $t=2k$.
Take any $m,n>t$ and $\mathcal{A}=J_{\binom{n-k}{k}\times(m-k)}\binom{[n-k]}{k}$ and $\mathcal{B}=\binom{[m-k]}{k}J_{\binom{m-k}{k}\times(n-k)}$. Clearly, $\mathcal{A}\subset\binom{[m+n-2k]}{m}$ and $\mathcal{B}\subset\binom{[m+n-2k]}{n}$ are cross-t-intersecting and for

any $x\in[m-k]$ $\deg_{\mathcal{B}} x=\binom{m-k-1}{k-1}=\frac{t|\mathcal{B}|}{2(m-k)}$, while for any $x\in[m-k+1,m+n-2k]$ $\deg_{\mathcal{A}} x=\binom{n-k-1}{k-1}=\frac{t|\mathcal{A}|}{2(n-k)}$.

Remark. For odd $t=2k-1>\frac{2m-1}{3}$ Theorem 3 is not true as the following example shows. Take $m<\frac{3t+1}{2}$ and any $n>t$. Set $\mathcal{A}=J_{\binom{n-k}{k-1}\times(m-k+1)}\binom{[n-k]}{k-1}$ and $\mathcal{B}=\binom{[m-k+1]}{k}J_{\binom{m-k+1}{k}\times(n-k)}$. Clearly, $\mathcal{A}\subset\binom{[m+n-2k+1]}{m}$ and $\mathcal{B}\subset\binom{[m+n-2k+1]}{n}$ are cross-t-intersecting and for any $x\in[m-k+1]$ $\deg_{\mathcal{B}} x=\binom{m-k}{k-1}<\frac{t|\mathcal{B}|}{2(m-k)}$, while for any $x\in[m-k+2,m+n-2k+1]$ $\deg_{\mathcal{A}} x=\binom{n-k-1}{k-2}<\frac{t|\mathcal{A}|}{2(n-k)}$. Using the ideas of the proof of Theorem 3 we can prove that in this case there is an element of the ground set s.t. either

$$\deg_{\mathcal{A}} x\geq\frac{t|\mathcal{A}|}{2(n-k)}\ \text{and}\ \deg_{\mathcal{B}} x\geq\frac{t|\mathcal{B}|}{2(m-k+1)}$$

or

$$\deg_{\mathcal{A}} x\geq\frac{t|\mathcal{A}|}{2(n-k+1)}\ \text{and}\ \deg_{\mathcal{B}} x\geq\frac{t|\mathcal{B}|}{2(m-k)}.$$

However, it can be shown that it is not tight. ∎

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